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Stochastic differential algebraic equations of index 1 and applications in circuit simulation

Renate Winkler*

Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, Berlin 10099, Germany

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Abstract

We discuss differential-algebraic equations driven by Gaussian white noise, which are assumed to have noise-free constraints and to be uniformly of DAE-index 1.

We first provide a rigorous mathematical foundation of the existence and uniqueness of strong solutions. Our theory is based upon the theory of stochastic differential equations (SDEs) and the theory of differential-algebraic equations (DAEs), to each of which our problem reduces on making appropriate simplifications.

We then consider discretization methods; implicit methods are necessary because of the differential-algebraic structure, and we consider adaptations of such methods used for SDEs. The consequences of an inexact solution of the implicit equations, roundoff and truncation errors, are analysed by means of the mean-square numerical stability of general drift-implicit discretization schemes for SDEs. We prove that the convergence properties of our drift-implicit Euler scheme, split-step backward Euler scheme, trapezoidal scheme and drift-implicit Milstein scheme carry over from the corresponding properties of these methods applied to SDEs.

Finally, we show how the theory applies to the transient noise simulation of electronic circuits.

Keywords: Stochastic differential equations; Differential algebraic equations; Numerical methods; Mean square numerical stability; Transient noise analysis; Circuit simulation

1. Introduction

In this paper we deal with Stochastic Differential Algebraic Equations (SDAEs) of the type

$$Ax'(t) + f(x(t),t) + G(x(t),t)\xi(t) = 0, \tag{1.1}$$

* Tel.: +49-30-2093-2360; fax: +49-30-2093-2232.
E-mail address: winkler@mathematik.hu-berlin.de (R. Winkler).

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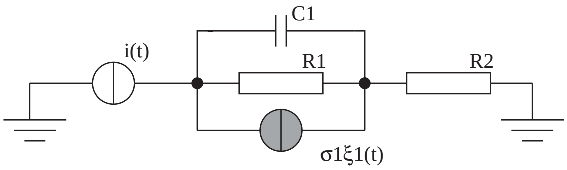

Fig. 1.

where f is a vector-valued function of dimension n, A is a constant singular $n \times n$ matrix with rank r, ξ stands for an m-dimensional vector of independent Gaussian white noise processes and G is an $n \times m$-dimensional matrix-function. For a mathematical treatment of (1.1) we understand it as a stochastic integral equation

$$Ax(s)|_{t_0}^{t} + \int_{t_0}^{t} f(x(s),s)\,\mathrm{d}s + \int_{t_0}^{t} G(x(s),s)\,\mathrm{d}w(s) = 0, \tag{1.2}$$

where the second integral is an Itô integral, and w denotes an m-dimensional Wiener process given on the probability space $(\Omega, \mathscr{F}, P)$ with a filtration $(\mathscr{F}_t)_{t \geqslant t_0}$.

A solution x is a vector-valued stochastic process of dimension n that depends both on the time t and an element ω of the probability space Ω. The argument ω is omitted in the notations above. The unknown $x(t) = x(t,\cdot)$ is a vector-valued random variable in $L^2(\Omega, \mathscr{F}, P; \mathbb{R}^n)$, and the identity in (1.2) means identity for all t and almost surely in ω. The short-hand notation (1.1) emphasizes the relations of (1.1) to its deterministic counterparts, but it may be misleading for readers who are less familiar with the stochastic background. Though the notation $x'(t)$ is used in (1.1), a typical realization $x(\cdot,\omega)$ of the solution is nowhere differentiable.

The application we have in mind is the transient noise simulation of electronic circuits. In the deterministic case the circuits are modelled by large, specially structured DAEs. Noise in the system is modelled by adding Gaussian white noise sources.

Here, we give a simple linear example with dimensions $n = 2$, $m = 1$:

$$\begin{pmatrix} c_1 & -c_1 \\ -c_1 & c_1 \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} g_1 & -g_1 \\ -g_1 & g_1 + g_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} -i(t) \\ 0 \end{pmatrix} + \begin{pmatrix} \sigma_1 \\ -\sigma_1 \end{pmatrix} \xi_1(t) = 0 \tag{1.3}$$

with parameters c_1, g_1, g_2, $\sigma_1 > 0$ and a given continuous scalar function i. The SDAE (1.3) serves as a mathematical model of the nodal potentials in the linear electrical network described in Fig. 1.

Now, we come back to our general problem (1.1). Due to the singularity of the matrix A the deterministic part of (1.1)

$$Ax'(t) + f(x(t),t) = 0, \tag{1.4}$$

where the solution x is a deterministic function of t, forms a DAE. Solutions have to fulfil the constraints of the equation. The solution components belonging to $\ker A$ (we call them the algebraic components) do not occur under the differential operator, and the inherent dynamics live only in a lower-dimensional subspace.

The deterministic part of our simple example (1.3) reads

$$\begin{pmatrix} c_1 & -c_1 \\ -c_1 & c_1 \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} g_1 & -g_1 \\ -g_1 & g_1+g_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} -i(t) \\ 0 \end{pmatrix} = 0. \tag{1.5}$$

The constraints are given only implicitly. Combining both rows one obtains the constraint $g_2 x_2(t) - i(t) = 0$. The solution component $x_2(\cdot)$ is as smooth as the input function $i(\cdot)$. If $i(\cdot)$ is not differentiable, neither is $x_2(\cdot)$. At first glance this seems to contradict the DAE (1.5). At a closer look one sees that only the difference $x_1(\cdot) - x_2(\cdot)$ has to be differentiable and a more exact formulation of (1.5) would be

$$\begin{pmatrix} c_1 & -c_1 \\ -c_1 & c_1 \end{pmatrix} \left(\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \right)' + \begin{pmatrix} g_1 & -g_1 \\ -g_1 & g_1+g_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} -i(t) \\ 0 \end{pmatrix} = 0.$$

The solution may be represented as the sum of differentiable and algebraic components in the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) - x_2(t) \\ 0 \end{pmatrix} + \begin{pmatrix} x_2(t) \\ x_2(t) \end{pmatrix}.$$

DAEs are usually classified by their index. The literature on DAEs contains a number of different definitions of this term pointing to different properties of the considered DAEs. Fortunately, they widely coincide in characterizing the special type of DAEs (1.4) to be of index 1. Let us sketch the possibly best-known definitions:

> The DAE (1.4) has differential index 1 iff differentiating the constraints once leads to an (implicit) ODE.
>
> The DAE (1.4) has perturbation index 1 iff perturbations in the solutions caused by perturbations in the right-hand side can be estimated by the latter, or, in other words, the solution does not depend on derivatives of input signals.
>
> The DAE (1.4) has tractability index 1 iff the constraints are locally solvable for the algebraic components.

We consider the tractability index definition to be most suitable for our purposes. It is the one with the lowest smoothness suppositions (e.g. it is applicable to the example of Eq. (1.5) with a continuous input function i), it allows to prove that (1.4) has perturbation index 1, too, and it is equivalent to the definition of the differential index 1 if the function f in (1.4) is sufficiently smooth.

If a DAE has tractability index 1, it involves a coupling of an integration task and a nonlinear equation solving task. If a DAE is of higher index, the constraints are not locally solvable for the algebraic components, and there exist solution components that are determined by a hidden differentiation step only. For a detailed analysis of DAEs and their numerics we refer to the monographs [2,5,12,14,16] or to the review papers [22,23,27].

SDAEs are a generalization of deterministic differential-algebraic equations (DAEs) as well as stochastic differential equations (SDEs). Much research has been devoted to the numerical solution of SDEs. Let us refer to the monographs [19,25] and the recent overview in [28], where more than 300 references are comprised. As an example of questions treated recently we refer to [6] in this journal. However, only first attempts have been made towards a numerical analysis of SDAEs: In [29,30] linear SDAEs are analysed and the convergence of the drift-implicit Euler scheme is proved. In [26] a scheme with strong order 1 is developed for the specially structured SDAEs that arise in transient noise simulation for electronic circuits. Later we will point out its relation to the drift-implicit Milstein scheme.

In the present paper we will prove the existence and uniqueness of solutions of general nonlinear SDAEs of index 1, and develop and analyse a number of numerical schemes for SDAEs. We put special emphasis on estimating the influence of computational as well as truncation errors. We use the mean-square numerical stability, which we prove for general drift-implicit discretization schemes for SDEs.

In Section 2 we define SDAEs of index 1 and present an analysis of such systems. We formulate initial value problems, prove the existence and uniqueness of strong solutions, and give estimates for the growth of solutions. This generalizes results in [29,30] to nonlinear systems.

In Section 3 we provide a proof of mean-square numerical stability for drift-implicit discretization methods for (ordinary) SDEs. Numerical stability allows to estimate the influence of computational and truncation errors in the discrete systems.

In Section 4 we present and discuss discretization schemes suitable for SDAEs, which are obtained by adapting drift-implicit schemes used for SDEs. In particular, we consider the drift-implicit Euler scheme, the split-step backward Euler scheme, the trapezoidal rule, and the drift-implicit Milstein scheme. Finally, in Section 5 we describe the SDAEs arising in transient noise simulation for electrical circuits. We give sufficient conditions for these systems to fulfil the assumptions made in the previous sections.

2. SDAEs of index 1

We consider the SDAE

$$Ax(s)|_{t_0}^{t} + \int_{t_0}^{t} f(x(s),s)\,\mathrm{d}s + \int_{t_0}^{t} G(x(s),s)\,\mathrm{d}w(s) = 0, \qquad t \in \mathscr{J}, \tag{2.1}$$

where A is a constant nonsingular matrix in $\mathbb{R}^{n\times n}$, $\mathscr{J}=[t_0,T]$, $f:\mathbb{R}^n\times\mathscr{J}\to\mathbb{R}^n$, $G:\mathbb{R}^n\times\mathscr{J}\to\mathbb{R}^{n\times m}$ are continuous functions, and, moreover, f possesses continuous derivatives with respect to x, w denotes an m-dimensional Wiener process given on the probability space $(\Omega,\mathscr{F},P)$ with a filtration $(\mathscr{F}_t)_{t\geqslant t_0}$. We are interested in strong solutions defined as follows:

Definition 1. A strong solution of (2.1) is a process $x(\cdot)=(x(t))_{t\in\mathscr{J}}$ with continuous sample paths that fulfils the following conditions:

- $x(\cdot)$ is adapted to the filtration $(\mathscr{F}_t)_{t\in\mathscr{J}}$,
- $\int_{t_0}^{t} |f_i(x(s),s)|\,\mathrm{d}s < \infty$ a.s., $\forall i=1,\dots,n$, $\forall t\in\mathscr{J}$,
  $\int_{t_0}^{t} g_{ij}^2(x(s),s)\,\mathrm{d}w(s) < \infty$ a.s., $\forall i=1,\dots,n$, $\forall j=1,\dots,m$, $\forall t\in\mathscr{J}$,
- (2.1) holds a.s.

In [29,30] it is shown that special conditions are needed to ensure solution processes that are not directly affected by white noise. Then the SDAEs are called SDAEs without direct noise, otherwise with direct noise. To avoid a solution process that is directly affected by white noise we have to assume that the noise sources do not appear in the constraints. This means that

$$\operatorname{im} G(x,t) \subseteq \operatorname{im} A \quad \forall (x,t) \in \mathbb{R}^n \times \mathcal{J}.$$

Further, we assume here that the deterministic part

$$Ax'(t) + f(x(t),t) = 0, \quad t \in \mathcal{J}, \tag{2.2}$$

is globally an index 1 DAE in the sense that the constraints are regularly and globally uniquely solvable for the algebraic variables, the components of x belonging to the kernel of the matrix A. The globally unique solvability is stronger than the tractability index 1 condition, which requires only the nonsingularity of the corresponding Jacobian and guarantees only local solvability of the constraints for the algebraic variables. Summarizing both assumptions we define:

Definition 2. The SDAE (2.1) is said to be an *index 1 SDAE* if

- the noise sources do not appear in the constraints, and
- the constraints are globally uniquely solvable for the algebraic variables.

To be more precise we will distinguish the differential and algebraic solution components as well as the constraints by means of the special projectors

$$Q \text{ onto } \ker A, \quad P := I - Q \text{ along } \ker A, \quad R \text{ along } \operatorname{im} A.$$

(A matrix Q is a projector iff $Q^2 = Q$. It projects onto its image and along its kernel.) Now we split the solution components into differential and algebraic components

$$x = Px + Qx =: u + v, \quad x \in \mathbb{R}^n, \; u \in \operatorname{im} P, \quad v \in \operatorname{im} Q$$

and, applying the projectors $(I - R)$ and R, the equations of the DAE (2.2) into differential ones and constraints:

$$Ax'(t) + (I - R) f(x(t),t) = 0, \tag{2.3}$$

$$R f(x(t),t) = 0. \tag{2.4}$$

Solving the constraints for the algebraic solution components means solving $Rf(u+v,t) = 0$, where $Av = 0$ for v, or, equivalently, solving

$$Av + Rf(u+v,t) = 0 \tag{2.5}$$

for v. We denote the solution by

$$v = \hat{v}(u,t). \tag{2.6}$$

With these notations we have the following characterization of index 1 SDAEs: The SDAE (2.1) is an index 1 SDAE if, for all $(x,t) \in \mathbb{R}^n \times \mathscr{J}$,

- $RG(x,t) = 0$, and
- the Jacobian $J(x,t) := A + Rf'_x(x,t)$ of (2.5) is nonsingular and the implicitly defined function $\hat{v}$ exists globally and uniquely.

The latter condition is guaranteed if the inverse $(A + Rf'_x(x,t))^{-1}$ is uniformly bounded. Furthermore, then also perturbed constraints

$$Rf(u+v,t) = Rd, \quad d \in \mathbb{R}^n,$$

are uniquely solvable for the algebraic variables. We will denote the solution by

$$v = \hat{v}(u,t,d) \quad \text{with the convention that } \hat{v}(u,t,0) = \hat{v}(u,t).$$

Next, let us point out that the above manipulations are applicable to the SDAE (2.1), too. This is due to the special structure of the considered SDAE, where the matrix A, and, hence, the projectors P, Q, R are constant. For any constant $n \times n$ matrix B we have

$$B \cdot \int_{t_0}^{t} G(x(s),s)\,\mathrm{d}w(s) = \int_{t_0}^{t} B \cdot G(x(s),s)\,\mathrm{d}w(s)$$

for the Itô-integral in (2.1). Applying R and $(I-R)$, using the identity $RA = 0$ and the condition $RG(x,t) = 0$, we are able to split (2.1) analogously to its deterministic counterpart into

$$Ax(s)|_{t_0}^{t} + \int_{t_0}^{t} (I-R)f(x(s),s)\,\mathrm{d}s + \int_{t_0}^{t} G(x(s),s)\,\mathrm{d}w(s) = 0 \tag{2.7}$$

$$\int_{t_0}^{t} Rf(x(s),s)\,\mathrm{d}s = 0. \tag{2.8}$$

A solution of (2.1) has to satisfy (2.8) for all $t \in \mathscr{J}$ almost surely. Since f is continuous and the solution paths have to be almost surely continuous we can conclude

$$Rf(x(t),t) = 0, \quad t \in \mathscr{J}, \text{ a.s.} \tag{2.9}$$

Solving these deterministic constraints for the algebraic variables we obtain

$$x(t) = Px(t) + Qx(t) := u(t) + \hat{v}(u(t),t). \tag{2.10}$$

Now, let us consider initial conditions to the SDAE (2.1). To find a solution to a given $\mathscr{F}_{t_0}$-measurable initial value x_0, this value has to fulfil the constraints at the initial time-point almost surely, i.e.,

$$Rf(x_0,t_0) = 0 \quad \text{a.s.} \tag{2.11}$$

The random variable x_0 is said to be a consistent initial value for the index 1 SDAE iff (2.11) is fulfilled. Eq. (2.11) represents n-rank A independent scalar conditions. There are only rank A free initial parameters. One way to determine a consistent initial value x_0 with given $Ax_0 := Ax^0$ for any $\mathscr{F}_{t_0}$-measurable $\mathbb{R}^n$-valued random variable x^0 is to solve the system

$$A(x_0 - x^0) = 0, \quad Rf(x_0,t_0) = 0,$$

or, equivalently,

$$A(x_0 - x^0) + Rf(x_0, t_0) = 0.$$

Using function (2.6) the solution of this system can be represented as

$$x_0 := Px_0 + Qx_0 := Px^0 + \hat{v}(Px^0, t_0).$$

In general, unless $Qx^0 = \hat{v}(Px^0, t_0)$, the consistent initial value x_0 will differ from the given value x^0. According to this setting initial value problems can be formulated as

$$A(x(t) - x^0) + \int_{t_0}^{t} f(x(s), s)\,\mathrm{d}s + \int_{t_0}^{t} G(x(s), s)\,\mathrm{d}w(s) = 0, \quad t \in \mathscr{J}, \tag{2.12}$$

or, abbreviated according to (1.1), as

$$Ax'(t) + f(x(t), t) + G(x(t), t)\xi(t) = 0, \quad t \in \mathscr{J}, \quad A(x(t_0) - x^0) = 0. \tag{2.13}$$

In this section we aim at the existence and uniqueness of solutions of (2.13) and related properties. The main tool will be a theoretical equivalence between (2.13) and a so-called inherent regular SDE. To find the inherent regular SDE we follow the steps:

- Applying the projectors P and Q along and onto $\ker A$ we split the solution vector into differential and algebraic components. We solve the constraints (2.9) for the algebraic components to obtain representation (2.10) for the solution:

$$x(t) = Px(t) + Qx(t) := u(t) + \hat{v}(u(t), t).$$

- We insert $v(t) = \hat{v}(u(t), t)$ into the stochastic differential equations (2.7) and use $Ax = Au$, $Ax^0 = APx^0$ to obtain

$$A(u(t) - Px^0) + \int_{t_0}^{t} (I - R)f(u + \hat{v}(u, s), s)\,\mathrm{d}s + \int_{t_0}^{t} G(u + \hat{v}(u, s), s)\,\mathrm{d}w(s) = 0. \tag{2.14}$$

- We scale the system by a suitable nonsingular matrix D such that $DA = P$. Then $A^- := D(I - R)$ is a pseudo-inverse with $A^-A = P$, $AA^- = (I - R)$. Using $Pu = u$, $u_0 := Px^0$ we obtain

$$u(t) - u_0 + \int_{t_0}^{t} \underbrace{A^- f(u + \hat{v}(u, s), s)}_{:=\hat{f}(u,s)}\,\mathrm{d}s + \int_{t_0}^{t} \underbrace{A^- G(u + \hat{v}(u, s), s)}_{:=\hat{G}(u,s)}\,\mathrm{d}w(s) = 0. \tag{2.15}$$

Eq. (2.15) is a regular SDE in the differential part u of the solution with $\operatorname{im} P$ as an invariant subspace. This becomes obvious by the following:
Since $DA = P$, we have $QDA = QP = 0$ and thus $QD(I - R) = 0$.
Applying the projector Q to (2.15) with the initial value $u(t_0)$ we obtain

$$Q(u(t) - u(t_0)) + 0 + 0 = 0, \quad t \in \mathscr{J}.$$

Hence, $Qu(t_0) = 0$ implies that $Qu(t) = 0$ for all $t \in \mathscr{J}$, or, in other words, $u(t_0) \in \operatorname{im} P$ implies that $u(t) \in \operatorname{im} P$ for all $t \in \mathscr{J}$.

Definition 3. Eq. (2.15) is called an *inherent regular SDE* of the SDAE (2.12).

The inherent regular SDE (2.15) together with the assembling of the solution (2.10) is equivalent to the original initial value problem for the SDAE (2.12).

Based on this fact we are now able to prove our main theorem on the existence and uniqueness of strong solutions of index 1 SDAEs:

Theorem 4. *Suppose that* (2.1) *is an index* 1 *SDAE and that the Jacobian* $J(x,t) := A + Rf_x'(x,t)$ *of* (2.5) *possesses a globally bounded inverse.*

Suppose that f *and* G *are globally Lipschitz-continuous with respect to* x, *continuous with respect to* t, *and that* Ax^0 *is* $\mathscr{F}_{t_0}$*-measurable, independent of the Wiener process* w, *and with finite second moments.*

Then there exists a solution process $x(\cdot)$ *of* (2.12) *that is pathwise unique. Moreover, the solution process* $x(\cdot)$ *is square-integrable and fulfils*

$$\mathbb{E}|x(t)|^2 \leqslant c_0(t) + c_1(1 + \mathbb{E}|Px_0|^2) \cdot \mathrm{e}^{c_2 L(t-t_0)}$$

with a continuous function $c_0(\cdot)$ (*resulting from the inhomogeneity in the constraints*), *and constants* c_1, c_2. *If*, *additionally*, *the function* Rf *is Lipschitz-continuous with respect to* t, *then there exist constants* c_3, c_4 *such that*

$$\mathbb{E}|x(t) - x_0|^2 \leqslant c_3(t - t_0)^2 + c_4(1 + \mathbb{E}|Px_0|^2) \cdot (t - t_0) \cdot \mathrm{e}^{c_2 L(t-t_0)}.$$

Proof. First, we note that, for continuous functions and compact time-intervals, the global Lipschitz continuity with respect to x implies the usual growth condition: One has

$$|f(x,t)| \leqslant \max(\|f(0,\cdot)\|_\infty, L_f)(1 + |x|) \quad \forall x \in \mathbb{R}^n, \quad \forall t \in \mathscr{J},$$

where $\|y(\cdot)\|_\infty := \max_{t\in\mathscr{J}} |y(t)|$ denotes the Chebychev-norm and L_f denotes the Lipschitz constant of f with respect to x.

An analogous argument applies to the matrix-valued function G.

Next, the implicit function $\hat{v}$ from (2.6) is globally Lipschitz-continuous, too. The function $\hat{v}$ solves

$$h(v; u, t) := Av + Rf(u + v, t) = 0.$$

Since the function h is continuously differentiable and Lipschitz-continuous with respect to v and u, also $\hat{v}$ is continuously differentiable with respect to u and

$$\begin{aligned} \hat{v}_u'(u,t) &= -h_v'(\hat{v}(u,t), u, t)^{-1} h_u'(\hat{v}(u,t), u, t) \\ &= -(A + Rf_x'(u + \hat{v}(u,t), t))^{-1} Rf_x'(u + \hat{v}(u,t), t). \end{aligned}$$

Since the Jacobian $h_v'(v,u,t) = J(u + v, t)$ is supposed to have a uniformly bounded inverse, i.e., $\|J(x,t)^{-1}\| \leqslant M$ holds with a uniform constant M, $\|\hat{v}_u'(u,t)\|$ is bounded by a constant $L_{\hat{v}} \leqslant M\|R\|L_f$. Hence, $\hat{v}$ is Lipschitz-continuous with respect to u with this constant. If, additionally, the function Rf is Lipschitz-continuous with respect to t, then $\hat{v}$ is Lipschitz-continuous with respect to t with a constant $L_{\hat{v},t}$. Considering the dependence of the function $\hat{v} = \hat{v}(u,t,d)$ on perturbations of the constraints Rd, $d \in \mathbb{R}^n$, we note that $\hat{v}$ is Lipschitz-continuous with respect to d with a constant $L_{\hat{v},d} = M\|R\|$.

Now, we have the conditions of the usual existence and uniqueness theorem for SDEs (see e.g. [1,18]) for the inherent regular SDE (2.15)):

- *Lipschitz condition*: The coefficient $\hat{f}$ with $\hat{f}(u,t) := A^- f(u+\hat{v}(u,t),t)$ is Lipschitz-continuous with respect to u with a constant $L_{\hat{f}} \leqslant \|A^-\| L_f(1+L_{\hat{v}})$. An analogous argument applies to $\hat{G}$.
- *Growth condition*: Since f and $\hat{v}$ depend continuously on t, also $\hat{f}$ depends continuously on t. Hence, the growth condition follows from the global Lipschitz condition. Again, an analogous argument applies to $\hat{G}$.
- *Initial condition*: Since Ax^0 is $\mathscr{F}_{t_0}$-measurable, independent of the Wiener process w, and has finite second moments, the same is true for $Px^0 = DAx^0$.

Applying the usual existence and uniqueness theorem for SDEs to the inherent regular SDE we obtain: The inherent regular SDE (2.15) has a pathwise unique continuous solution process u that is square-integrable and fulfils

$$\mathbb{E}|u(t)|^2 \leqslant (1+\mathbb{E}|u_0|^2)\cdot \mathrm{e}^{c_5\hat{L}(t-t_0)}, \quad \forall t \in \mathscr{J}, \tag{2.16}$$

$$\mathbb{E}|u(t)-u_0|^2 \leqslant c_6(1+\mathbb{E}|u_0|^2)\cdot (t-t_0)\cdot \mathrm{e}^{c_5\hat{L}(t-t_0)}, \quad \forall t \in \mathscr{J}, \tag{2.17}$$

with constants c_5, c_6, where $\hat{L}$ is a Lipschitz constant for the functions $\hat{f}$, $\hat{G}$. Assembling the solution x of the SDAE (2.1) as in (2.10)

$$x(t) := u(t) + \hat{v}(u(t),t), \quad t \in \mathscr{J}$$

gives a pathwise unique continuous solution process of the original SDAE. Due to the Lipschitz and continuity properties of the implicit function $\hat{v}$ we can estimate:

$$|x(t)| \leqslant |u(t)| + |\hat{v}(u(t),t) - \hat{v}(0,t)| + |\hat{v}(0,t)| \leqslant (1+L_{\hat{v}})|u(t)| + |\hat{v}(0,t)|.$$

Hence, the solution $x(t)$ is also square integrable for all $t \in \mathscr{J}$, and

$$\begin{aligned}
\mathbb{E}|x(t)|^2 &\leqslant \mathbb{E}((1+L_{\hat{v}})|u(t)| + |\hat{v}(0,t)|)^2 \\
&\leqslant 2|\hat{v}(0,t)|^2 + 2(1+L_{\hat{v}})^2\mathbb{E}|u(t)|^2 \\
&\leqslant 2|\hat{v}(0,t)|^2 + 2(1+L_{\hat{v}})^2(1+\mathbb{E}|u_0|^2)\cdot \mathrm{e}^{c_5\hat{L}(t-t_0)} \\
&=: c_0(t) + c_1(1+\mathbb{E}|Px_0|^2)\cdot \mathrm{e}^{c_2L(t-t_0)},
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}|x(t)-x_0|^2 &= \mathbb{E}|u(t)+\hat{v}(u(t),t) - (u_0+\hat{v}(u_0,t_0))|^2 \\
&\leqslant \mathbb{E}((1+L_{\hat{v}})|u(t)-u_0| + L_{\hat{v},t}|t-t_0|)^2 \\
&\leqslant 2L_{\hat{v},t}^2(t-t_0)^2 + (1+L_{\hat{v}})^2 c_6(1+\mathbb{E}|u_0|^2)(t-t_0)\mathrm{e}^{c_5\hat{L}(t-t_0)} \\
&=: c_3(t-t_0)^2 + c_4(1+\mathbb{E}|Px_0|^2)(t-t_0)\mathrm{e}^{c_2L(t-t_0)}. \qquad \square
\end{aligned}$$

3. Numerical stability, consistency and mean-square convergence for discretization methods for SDEs

The numerical treatment of SDAEs incorporates not only truncation errors and roundoff errors, but also defects in solving the constraints or in solving the nonlinear equations in drift-implicit methods. It is not appropriate to assume that these errors tend to zero if the stepsizes do so.

Analogously to the analysis of SDAEs in Section 2 we will trace back the properties of certain discretization schemes for SDAEs to those for SDEs. Although there is a well-developed convergence analysis for discretization schemes for SDEs, less emphasis has been put on a numerical stability analysis to estimate the effect of errors. Therefore, we supplement the known convergence results by a theorem concerning the numerical stability of discretization schemes for SDEs.

Various stability properties of numerical methods are discussed in the literature (see e.g. [7,15,20]). Numerical stability for a discretization scheme means that differences of the discrete solutions due to errors on the right-hand side of the discrete system can be estimated by the maximum of these errors multiplied by a grid-independent stability constant. Numerical stability allows to conclude convergence from consistency. In order to distinguish this stability concept from others, it is sometimes called zero stability. It should not be mistaken for properties like asymptotic stability, which guarantee that for fixed stepsizes (and long or unbounded time intervals) qualitative properties of the exact solutions like damping behaviour in dissipative systems are preserved by the discrete approximations.

We aim at a numerical stability inequality for discretization schemes for SDEs concerning the mean-square norm of errors of the discrete solution. We will estimate them by the mean-square norm as well as the conditional mean of the errors perturbing the right-hand sides. This phenomenon is already known from the convergence proofs, e.g. in [3,25]. In [11,19] it occurs only implicitly due to the comparison with the truncated Itô–Taylor expansions.

We denote the mean-square norm of a vector-valued square-integrable random variable $z \in L_2(\Omega, \mathbb{R}^n)$ by

$$\|z\|_{L_2} := (\mathbb{E}|z|^2)^{1/2}.$$

Let us consider the initial value problem for the SDE

$$x(s)|_{t_0}^{t} + \int_{t_0}^{t} f(x(s),s)\,\mathrm{d}s + \int_{t_0}^{t} G(x(s),s)\,\mathrm{d}w(s) = 0, \quad t \in \mathscr{J},\ x(t_0) = x_0, \tag{3.1}$$

where $\mathscr{J} = [t_0, T], f : \mathbb{R}^n \times \mathscr{J} \to \mathbb{R}^n, G : \mathbb{R}^n \times \mathscr{J} \to \mathbb{R}^{n\times m}, w$ is an m-dimensional Wiener process on the given probability space $(\Omega, \mathscr{F}, P)$ with a filtration $(\mathscr{F}_t)_{t\in\mathscr{J}}$, and x_0 is a given $\mathscr{F}_{t_0}$-measurable initial value, independent of the Wiener process and with finite second moments. We assume that there exists a pathwise unique strong solution $x(\cdot)$.

Moreover, let us consider a generally drift-implicit numerical scheme of the form

$$x_\ell = x_{\ell-1} + \varphi(x_{\ell-1}, x_\ell; t_{\ell-1}, h_\ell) + \psi(x_{\ell-1}; t_{\ell-1}, h_\ell, I_\ell), \quad \ell = 1, \ldots, N, \tag{3.2}$$

on the deterministic grid $t_0 < t_1 < \cdots < t_N = T$ with stepsizes $h_\ell := t_\ell - t_{\ell-1}$, and a collection of multiple stochastic integrals $I_\ell = I_{t_{\ell-1},h_\ell}$, $\ell = 1, \ldots, N$. Here, $I_{t,h}$ denotes a collection of M multiple stochastic integrals

$$I_{i_1,\ldots,i_k;t,h} = \int_t^{t+h} \mathrm{d}w_{i_k}(s_1) \int_t^{s_1} \mathrm{d}w_{i_{k-1}}(s_2) \cdots \int_t^{s_{k-1}} \mathrm{d}w_{i_1}(s_k),$$

where the indices $i_1,\dots,i_k$ are in $\{0,1,\dots,m\}$, k is bounded by certain finite order $k_{\max}$, and $\mathrm{d}w_0(s)$ corresponds to $\mathrm{d}s$.

For example, for the drift-implicit Euler scheme

$$x_\ell := x_{\ell-1} + h_\ell f(x_\ell, t_\ell) + G(x_{\ell-1}, t_{\ell-1})\Delta w_\ell, \quad \ell = 1,\dots,N,$$

where $\Delta w_\ell := (w(t_\ell) - w(t_{\ell-1})) = (I_{i;t_{\ell-1},h_\ell})_{i=1}^m$, one has $k_{\max} = 1$, $M = m$, and

$$\varphi(z,x;t,h) := hf(x,t+h), \quad \psi(z;t,h,y) := G(z,t)y,$$

for $z,x \in \mathbb{R}^n, t, t+h \in \mathscr{J}, y \in \mathbb{R}^m$. A similar setting for explicit schemes is used in [3]. We now formulate and prove our main theorem on numerical stability.

Theorem 5. *Assume that scheme (3.2) for the SDE (3.1) satisfies the following properties:*

- *For all* z, $\tilde{z}$, x, $\tilde{x} \in \mathbb{R}^n, h \in (0,h^1], t, t+h \in \mathscr{J}$ *we have:*

$$\text{(A1)} \quad |\varphi(z,x;t,h) - \varphi(\tilde{z},\tilde{x};t,h)| \leqslant hL_1|z-\tilde{z}| + hL_2|x-\tilde{x}|$$

  for some positive constants h^1, L_1, L_2.
- *For all* $h \in (0,h^1], t, t+h \in \mathscr{J}$ *and all* $\mathscr{F}_t$*-measurable random variables* $z_t, \tilde{z}_t$ *we have:*

$$\text{(A2)} \quad \mathbb{E}(\psi(z_t;t,h,I_{t,h}) - \psi(\tilde{z}_t;t,h,I_{t,h})|\mathscr{F}_t) = 0,$$

$$\text{(A3)} \quad \mathbb{E}(|\psi(z_t;t,h,I_{t,h}) - \psi(\tilde{z}_t;t,h,I_{t,h})|^2|\mathscr{F}_t) \leqslant hL_3^2|z_t - \tilde{z}_t|^2,$$

$$\text{(A4)} \quad \mathbb{E}|\psi(0;t,h,I_{t,h})|^2 < \infty.$$

  for some positive constant L_3.

Then there exists constants $a \geqslant 1$, $h^0 > 0$ *and a stability constant* $S > 0$ *such that the following holds true for each grid* $\{t_0,t_1,\dots,t_N\}$ *having the property* $h := \max_{\ell=1,\dots,N} h_\ell \leqslant h^0$ *and* $h \cdot N \leqslant a \cdot (T - t_0)$:

For all $\mathscr{F}_{t_0}$*-measurable, square-integrable initial values* $\tilde{x}_0$, x_0^*, *for all* $\ell \in \{1,\dots,N\}$ *and all* $\mathscr{F}_{t_\ell}$*-measurable perturbations* $\tilde{d}_\ell$, d_ℓ^* *having finite second moments the perturbed discrete systems*

$$\tilde{x}_\ell = \tilde{x}_{\ell-1} + \varphi(\tilde{x}_{\ell-1},\tilde{x}_\ell;t_{\ell-1},h_\ell) + \psi(\tilde{x}_{\ell-1};t_{\ell-1},h_\ell,I_{t_{\ell-1},h_\ell}) + \tilde{d}_\ell, \tag{3.3}$$

$$x_\ell^* = x_{\ell-1}^* + \varphi(x_{\ell-1}^*,x_\ell^*;t_{\ell-1},h_\ell) + \psi(x_{\ell-1}^*;t_{\ell-1},h_\ell,I_{t_{\ell-1},h_\ell}) + d_\ell^*, \tag{3.4}$$

$\ell = 1,\dots,N$, *have unique solutions* $\{\tilde{x}_\ell\}_{\ell=0}^N$, $\{x_\ell^*\}_{\ell=0}^N$, *and the mean-square norm* $\varepsilon_\ell := \|x_\ell^* - \tilde{x}_\ell\|_{L_2}$ *of their differences can be estimated by*

$$\max_{\ell=1,\dots,N} \varepsilon_\ell \leqslant S\left\{\varepsilon_0 + \max_{\ell=1,\dots,N}(\|s_\ell\|_{L_2}h^{-1/2} + \|r_\ell\|_{L_2}h^{-1})\right\}, \tag{3.5}$$

where $d_\ell := d_\ell^* - \tilde{d}_\ell$ *is splitted such that* $d_\ell = s_\ell + r_\ell$ *with* $\mathbb{E}(s_\ell|\mathscr{F}_{t_{\ell-1}}) = 0$, *or*

$$\max_{\ell=1,\dots,N} \varepsilon_\ell \leqslant S\left\{\varepsilon_0 + \max_{\ell=1,\dots,N}(\|d_\ell\|_{L_2}h^{-1/2} + \|\bar{d}_\ell\|_{L_2}h^{-1})\right\}, \tag{3.6}$$

where $\bar{d}_\ell := \mathbb{E}(d_\ell|\mathscr{F}_{t_\ell-1})$.

Definition 6. If scheme (3.2) for the SDE (3.1) fulfils the assertion of Theorem 5, we call it *numerically stable in the mean-square sense*.

For the proof of Theorem 5 we need a discrete analogue of Gronwall's inequality.

Lemma 7. *Let $a_\ell, \ell = 1, \dots, N$, and C_1, C_2 be nonnegative real numbers and assume that the inequalities*

$$a_\ell \leqslant C_1 + C_2 \frac{1}{N} \sum_{i=1}^{\ell-1} a_i, \quad \ell = 1, \dots, N,$$

are valid. Then we have $\max_{\ell=1,\dots,N} a_\ell \leqslant C_1 \exp(C_2)$.

Proof of Theorem 5. The proof is organized in three parts. First, we show the existence of unique solutions of the perturbed discrete systems. Second, we show that the second moments of these solutions exist, and, third, we derive a stability inequality.

Part 1 (*Existence of solutions $\tilde{x}_\ell$*): We consider scheme (3.3). If the function φ does not depend on the variable x, the right-hand side of (3.3) gives the new iterate $\tilde{x}_\ell$ explicitly. Otherwise, the new iterate $\tilde{x}_\ell$ is given by (3.3) only implicitly as the solution of the fixed point equation

$$x = \varphi(\tilde{x}_{\ell-1}, x; t_{\ell-1}, h_\ell) + \tilde{b}_\ell =: \eta_\ell(x; \tilde{x}_{\ell-1}, \tilde{b}_\ell), \tag{3.7}$$

where $\tilde{b}_\ell := \tilde{x}_{\ell-1} + \psi(\tilde{x}_{\ell-1}; t_{\ell-1}, h_\ell, I_\ell) + \tilde{d}_\ell$ is a known $\mathscr{F}_{t_\ell}$-measurable random variable. The function $\eta_\ell(x; z, b)$ is globally contractive with respect to x, since, due to the global Lipschitz condition (A1),

$$|\eta_\ell(x; z, b) - \eta_\ell(\tilde{x}; z, b)| \leqslant h_\ell L_2 |x - \tilde{x}| \leqslant \frac{1}{2}|x - \tilde{x}| \quad \forall h_\ell \leqslant h \leqslant h^0 \leqslant \frac{1}{2L_2}.$$

Thus, $\eta_\ell(\cdot; z, b)$ has a globally unique fixed point $x = \xi_\ell(z, b)$, and $\xi_\ell(\tilde{x}_{\ell-1}, \tilde{b}_\ell)$ gives the unique solution $\tilde{x}_\ell$ of (3.3). Moreover, $\xi_\ell(z, b)$ depends Lipschitz-continuously on z and b since

$$\begin{aligned}
|\xi_\ell(z, b) - \xi_\ell(\tilde{z}, \tilde{b})| &= |\eta_\ell(\xi_\ell(z, b); z, b) - \eta_\ell(\xi_\ell(\tilde{z}, \tilde{b}); \tilde{z}, \tilde{b})| \\
&= |\varphi_\ell(z, \xi_\ell(z, b)) - \varphi_\ell(\tilde{z}, \xi_\ell(\tilde{z}, \tilde{b})) + b - \tilde{b}| \\
&\overset{A_1}{\leqslant} h_\ell L_1 |z - \tilde{z}| + h_\ell L_2 |\xi_\ell(z, b) - \xi_\ell(\tilde{z}, \tilde{b})| + |b - \tilde{b}| \\
&\leqslant h L_1 |z - \tilde{z}| + \frac{1}{2}|\xi_\ell(z, b) - \xi_\ell(\tilde{z}, \tilde{b})| + |b - \tilde{b}|,
\end{aligned}$$

$$|\xi_\ell(z, b) - \xi_\ell(\tilde{z}, \tilde{b})| \leqslant 2hL_1 |z - \tilde{z}| + 2|b - \tilde{b}|.$$

Part 2 (*Existence of finite second moments $\mathbb{E}|\tilde{x}_\ell|^2 < \infty$*): Assume that $\mathbb{E}|\tilde{x}_{\ell-1}|^2 < \infty$. We compare $\tilde{x}_\ell = \xi_\ell(\tilde{x}_{\ell-1}, \tilde{b}_\ell)$ with the deterministic value $x_\ell^0 := \xi_\ell(0, 0)$. Using the Lipschitz continuity of the implicit function ξ_ℓ we obtain

$$|\tilde{x}_\ell - x_\ell^0| = |\xi_\ell(\tilde{x}_{\ell-1}, \tilde{b}_\ell) - \xi_\ell(0, 0)| \leqslant 2hL_1 |\tilde{x}_{\ell-1}| + 2|\tilde{b}_\ell|,$$

$$\|\tilde{x}_\ell\|_{L_2} \leqslant \|x_\ell - x_\ell^0\|_{L_2} + \|x_\ell^0\|_{L_2} \leqslant 2hL_1 \|\tilde{x}_{\ell-1}\|_{L_2} + 2\|\tilde{b}_\ell\|_{L_2} + \|x_\ell^0\|_{L_2}.$$

It remains to show that $\|\tilde{b}_\ell\|_{L_2} < \infty$, which follows from

$$\|\psi(\tilde{x}_{\ell-1}; t_{\ell-1}, h_\ell, I_\ell)\|_{L_2} \overset{A_3}{\leqslant} h_\ell^{1/2} L_3 \|\tilde{x}_{\ell-1}\|_{L_2} + \|\psi(0; t_{\ell-1}, h_\ell, I_\ell)\|_{L_2} \overset{A_4}{<} \infty.$$

Part 3 (*Stability inequality*): Let x_ℓ^* and $\tilde{x}_\ell$, $\ell = 1, \dots, N$, be the unique solutions of the perturbed discrete systems (3.3), (3.4). We introduce the following notations for $i = 1, \dots, N$:

$$e_i := x_i^* - \tilde{x}_i, \quad \Delta\varphi_i := \varphi(x_{i-1}^*, x_i^*; t_{i-1}, h_i) - \varphi(\tilde{x}_{i-1}, \tilde{x}_i; t_{i-1}, h_i),$$

$$d_i := d_i^* - \tilde{d}_i, \quad \Delta\psi_i := \psi(x_{i-1}^*, t_{i-1}, h_i, I_{t_{i-1},h_i}) - \psi(\tilde{x}_{i-1}; t_{i-1}, h_i, I_{t_{i-1},h_i}),$$

and obtain from (3.3), (3.4) and Hölder's inequality that

$$e_\ell = e_{\ell-1} + \Delta\varphi_\ell + \Delta\psi_\ell + d_\ell = e_0 + \sum_{i=1}^{\ell} \Delta\varphi_i + \sum_{i=1}^{\ell} \Delta\psi_i + \sum_{i=1}^{\ell} d_i,$$

$$\mathbb{E}|e_\ell|^2 \leqslant 4 \left\{ \mathbb{E}|e_0|^2 + \mathbb{E}\left|\sum_{i=1}^{\ell} \Delta\varphi_i\right|^2 + \mathbb{E}\left|\sum_{i=1}^{\ell} \Delta\psi_i\right|^2 + \mathbb{E}\left|\sum_{i=1}^{\ell} d_i\right|^2 \right\}$$

holds for each $\ell = 1, \dots, N$. For the second summand on the right-hand side of the latter estimate we continue by using (A1) and $\ell h \leqslant Nh \leqslant T_a := a(T - t_0)$:

$$\left|\sum_{i=1}^{\ell} \Delta\varphi_i\right|^2 \leqslant \ell \sum_{i=1}^{\ell} |\Delta\varphi_i|^2 \overset{A1}{\leqslant} 2\ell h^2 \sum_{i=1}^{\ell} (L_1^2 |e_{i-1}|^2 + L_2^2 |e_i|^2)$$

$$\leqslant 2\ell h^2 (L_2^2 |e_\ell|^2 + (L_1^2 + L_2^2) \sum_{k=0}^{\ell-1} |e_i|^2) \leqslant \hat{L}_2 |e_\ell|^2 + \hat{L}_1 \frac{1}{N} \sum_{i=0}^{\ell-1} |e_i|^2,$$

$$\mathbb{E}\left|\sum_{i=1}^{\ell} \Delta\varphi_i\right|^2 \leqslant \hat{L}_2 \mathbb{E}|e_\ell|^2 + \hat{L}_1 \frac{1}{N} \sum_{i=0}^{\ell-1} \mathbb{E}|e_i|^2,$$

where $\hat{L}_1 := 2T_a^2(L_1^2 + L_2^2)$, $\hat{L}_2 := 2T_a L_2^2 h$. Using conditions (A2), (A3), and $Nh \leqslant T_a$, we estimate the third summand:

$$\mathbb{E}\left|\sum_{i=1}^{\ell} \Delta\psi_i\right|^2 = \mathbb{E}\left|\sum_{i,j=1}^{\ell} \Delta\psi_i^T \Delta\psi_j\right| \leqslant \sum_{i,j=1}^{\ell} \mathbb{E}|\Delta\psi_i^T \Delta\psi_j| \overset{A2}{=} \sum_{i=1}^{\ell} \mathbb{E}|\Delta\psi_i|^2$$

$$= \sum_{i=1}^{\ell} \mathbb{E}\mathbb{E}(|\Delta\psi_i|^2 | \mathscr{F}_{t_{i-1}}) \overset{A3}{\leqslant} h L_3^2 \sum_{i=0}^{\ell-1} \mathbb{E}|e_i|^2 \leqslant \hat{L}_3 \frac{1}{N} \sum_{i=0}^{\ell-1} \mathbb{E}|e_i|^2,$$

where $\hat{L}_3 := L_3^2 T_a$. We arrive, altogether, at the estimate

$$\mathbb{E}|e_\ell|^2 \leqslant 4 \left\{ \mathbb{E}|e_0|^2 + \hat{L}_2 \mathbb{E}|e_\ell|^2 + (\hat{L}_1 + \hat{L}_3) \frac{1}{N} \sum_{i=0}^{\ell-1} \mathbb{E}|e_i|^2 + \mathbb{E}\left|\sum_{i=0}^{\ell} d_i\right|^2 \right\}$$

for $\ell = 1,\ldots,N$. If necessary, we choose h^0 smaller such that $4\hat{L}_2 \leqslant \frac{1}{2}$ holds if $h < h^0$. We conclude that

$$\mathbb{E}|e_\ell|^2 \leqslant 8\left\{\mathbb{E}|e_0|^2 + (\hat{L}_1 + \hat{L}_3)\,\frac{1}{N}\sum_{i=0}^{\ell-1}\mathbb{E}|e_i|^2 + \mathbb{E}\left|\sum_{i=1}^{\ell} d_i\right|^2\right\}, \quad \ell = 1,\ldots,N.$$

We apply Lemma 7 to $a_\ell := \mathbb{E}|e_\ell|^2$, $\ell = 1,\ldots,N$, and obtain the semifinal estimate

$$\max_{\ell=1,\ldots,N}\mathbb{E}|e_\ell|^2 \leqslant \hat{S}\left\{\mathbb{E}|e_0|^2 + \max_{\ell=1,\ldots,N}\mathbb{E}\left|\sum_{i=1}^{\ell} d_i\right|^2\right\},$$

with $\hat{S} := 8\exp(8(\hat{L}_2 + \hat{L}_3))$. It remains to decompose the perturbation difference d_i into $d_i = s_i + r_i$ with $\mathbb{E}(s_i|\mathscr{F}_{t_{i-1}}) = 0$ for $i = 1,\ldots,N$. Then $\mathbb{E}s_i^T s_j = 0$ for $i \neq j$, and

$$\begin{aligned}\mathbb{E}\left|\sum_{i=1}^{\ell} d_i\right|^2 &\leqslant 2\mathbb{E}\left|\sum_{i=1}^{\ell} s_i\right|^2 + 2\mathbb{E}\left|\sum_{i=1}^{\ell} r_i\right|^2 = 2\sum_{i=1}^{\ell}\mathbb{E}|s_i|^2 + 2\mathbb{E}\left|\sum_{i=1}^{\ell} r_i\right|^2\\ &\leqslant 2\sum_{i=1}^{\ell}\mathbb{E}|s_i|^2 + 2\ell\sum_{i=1}^{\ell}\mathbb{E}|r_i|^2 \leqslant 2\left(\sum_{i=1}^{N}\mathbb{E}|s_i|^2 + \mathbb{E}|r_i|^2\,\frac{T_a}{h}\right)\\ &\leqslant 2\max_{i=1,\ldots,N}\left(\mathbb{E}|s_i|^2 + \mathbb{E}|r_i|^2\,\frac{T_a}{h}\right)\frac{T_a}{h}.\end{aligned}$$

Summarizing, we obtain

$$\max_{\ell=1,\ldots,N}\mathbb{E}|e_\ell|^2 \leqslant \hat{S}\left\{\mathbb{E}|e_0|^2 + 2\max_{\ell=1,\ldots,N}\left(\mathbb{E}|s_\ell|^2\,\frac{T_a}{h} + \mathbb{E}|r_\ell|^2\,\frac{T_a^2}{h^2}\right)\right\}.$$

Extracting the square root leads to

$$\max_{\ell=1,\ldots,N}\|e_\ell\|_{L_2} \leqslant \sqrt{2\hat{S}}\left\{\|e_0\|_{L_2} + \max_{\ell=1,\ldots,N}\left(\|s_\ell\|_{L_2}\sqrt{\frac{T_a}{h}} + \|r_\ell\|_{L_2}\,\frac{T_a}{h}\right)\right\}, \tag{3.8}$$

which yields the final estimate (3.5) with $S := \sqrt{2\hat{S}}\max(1,T_a)$. The estimate (3.6) follows for the particular splitting

$$d_\ell = s_\ell + r_\ell := (d_\ell - \bar{d}_\ell) + \bar{d}_\ell, \quad \bar{d}_\ell := \mathbb{E}(d_\ell|\mathscr{F}_{t_{\ell-1}}),$$

since

$$\|d_\ell - \bar{d}_\ell\|_{L_2}^2 \leqslant \mathbb{E}|(d_\ell - \bar{d}_\ell)|^2 + \mathbb{E}|\bar{d}_\ell|^2 = \mathbb{E}|(d_\ell - \bar{d}_\ell) + \bar{d}_\ell|^2 = \|d_\ell\|_{L_2}^2. \qquad \square$$

Applying Theorem 5 to local discretization errors provides convergence results. Now, we give the precise notions of strong (mean-square) consistency and strong (mean-square) convergence.

Definition 8. We call the numerical scheme (3.2) for the SDE (3.1) *strongly* (*mean-square*) *consistent* with order $\gamma > 0$ if the local error

$$l_\ell := x(t_\ell) - \{x(t_{\ell-1}) + \varphi(x(t_{\ell-1}),x(t_\ell);t_{\ell-1},h_\ell) + \psi(x(t_{\ell-1});t_{\ell-1},h_\ell,I_\ell)\}$$

satisfies

$$\|l_\ell\|_{L_2} \leqslant c \cdot h_\ell^{\gamma+\frac{1}{2}}, \quad \text{and} \quad \|\mathbb{E}(l_\ell \,|\, \mathscr{F}_{t_{\ell-1}})\|_{L_2} \leqslant \bar{c} \cdot h_\ell^{\gamma+1}, \quad \ell = 1, \ldots, N,$$

with constants c, $\bar{c} > 0$ only depending on the SDE and its solution.

We call the numerical scheme (3.2) for the SDE (3.1) *strongly* (*mean-square*) *convergent* with order $\gamma > 0$ if the global error $x(t_\ell) - x_\ell$ satisfies

$$\max_{\ell=1,\ldots,N} \|x(t_\ell) - x_\ell\|_{L_2} \leqslant C \cdot h^\gamma, \text{ where } h := \max_{\ell=1,\ldots,N} h_\ell,$$

with a grid-independent constant $C > 0$.

With these notions a scheme that is numerically stable in the mean-square sense and strongly (mean-square) consistent is strongly (mean-square) convergent. As a corollary from Theorem 5 we have:

Theorem 9. *If the numerical scheme* (3.2) *for the SDE* (3.1) *is strongly* (*mean-square*) *consistent with order* $\gamma > 0$ *and the assumptions of Theorem* 5 *hold, then* (3.2) *is strongly* (*mean-square*) *convergent with order* γ. *For the difference of the analytical solution* $x(t_\ell)$ *at the discrete time points and the solution of the perturbed numerical scheme* $\tilde{x}_\ell$ *we have the estimate*

$$\max_{\ell=1,\ldots,N} \|x(t_\ell) - \tilde{x}_\ell\|_{L_2} \leqslant S\left((c+\bar{c})h^\gamma + \max_{\ell=1,\ldots,N}(\tilde{\delta}_\ell/h^{1/2} + \bar{\tilde{\delta}}_\ell/h)\right), \tag{3.9}$$

where $\tilde{\delta}_\ell := \|\tilde{d}_\ell\|_{L_2}$, $\bar{\tilde{\delta}}_\ell := \|\mathbb{E}(\tilde{d}_\ell|\mathscr{F}_{t_{\ell-1}})\|_{L_2}$, *with* $\tilde{d}_\ell$ *from* (3.3).

Proof. The assertion follows by applying the triangle inequality

$$\max_{\ell=1,\ldots,N} \|x(t_\ell) - \tilde{x}_\ell\|_{L_2} \leqslant \max_{\ell=1,\ldots,N} \|x(t_\ell) - x_\ell\|_{L_2} + \max_{\ell=1,\ldots,N} \|x_\ell - \tilde{x}_\ell\|_{L_2}$$

and stability estimate (3.6) once to $\{x(t_\ell), x_\ell\}$ related to the perturbations $\{l_\ell, 0\}$, and once again to $\{x_\ell, \tilde{x}_\ell\}$ related to the perturbations $\{0, \tilde{d}_\ell\}$.

The strong (mean-square) convergence follows as a special case of (3.9) for $\tilde{d}_\ell = 0$. □

These general results apply rather easily to well-known schemes for SDEs. We illustrate this for the family of drift-implicit Euler schemes

$$x_\ell = x_{\ell-1} + h_\ell(\alpha f(x_\ell, t_\ell) + (1-\alpha) f(x_{\ell-1}, t_{\ell-1})) + G(x_{\ell-1}, t_{\ell-1})\Delta w_\ell, \tag{3.10}$$

and for the family of drift-implicit Milstein schemes

$$x_\ell = x_{\ell-1} + h_\ell(\alpha f(x_\ell, t_\ell) + (1-\alpha) f(x_{\ell-1}, t_{\ell-1})) + G(x_{\ell-1}, t_{\ell-1})\Delta w_\ell + \sum_{j=1}^{m} (g'_{j_x} G)(x_{\ell-1}, t_{\ell-1}) I_{(j),\ell}, \tag{3.11}$$

where $\alpha \in [0, 1]$ is a parameter, $\Delta w_\ell := (w(t_\ell) - w(t_{\ell-1})) = (I_{i;t_{\ell-1},h_\ell})_{i=1}^m$, g_j denotes the jth column of G, and $I_{(j),\ell} = (I_{j,i;t_{\ell-1},h_\ell})_{i=1}^m$ denotes the double Itô-integrals.

With $k_{\max}=1$, $M=m$, and

$$\varphi(z,x;t,h):=h(\alpha f(x,t+h)+(1-\alpha)f(z,t)), \quad \psi(z;t,h,y):=G(z,t)y,$$

for the family of Euler schemes (3.10), and the same function φ, $k_{\max}=2$, $M=m+m^2$, and

$$\psi(z;t,h,y,y_1,\ldots,y_m)=G(z,t)y+\sum_{j=1}^{m}((g_j)'_x\cdot G)(z,t)y_j,$$

for the family of Milstein schemes (3.11), both schemes have form (3.2). Checking the suppositions of Theorem 5 we see that these methods are numerically stable in the mean-square sense: (A1) follows from the Lipschitz continuity of the drift coefficient f, (A2) holds due to the explicit, nonanticipative discretization of the diffusion term, (A3) follows from the Lipschitz continuity of the diffusion coefficient G (and in case of the Milstein scheme of the functions $(g_j)'_x\cdot G$), and the more technical condition (A4) holds true due to the boundedness of the function $G(0,\cdot)$ (and the functions $(g_j)'_x\cdot G(0,\cdot)$) on the compact interval $\mathscr{J}$. Summarization we obtain:

Proposition 10. *Let the functions f and G be Lipschitz-continuous with respect to x. Then the Euler schemes (3.10) are numerically stable in the mean-square sense. If, additionally, the partial derivatives $(g_j)'_x, j=1,\ldots,m$, exist, and the functions $(g_j)'_x\cdot G$ are Lipschitz-continuous with respect to x, then the Milstein schemes (3.11) are numerically stable in the mean-square sense.*

From the literature (see e.g. [25]) it is known that the Euler schemes (3.10) are strongly consistent with order $\frac{1}{2}$ if, additionally, the coefficients are Hölder-continuous with exponent $\frac{1}{2}$ with respect to t. The Milstein schemes are strongly consistent with order 1 if the functions f,G are sufficiently smooth. Applying Theorem 9 gives the known strong (mean-square) convergence of the Euler schemes with order $\frac{1}{2}$ and the Milstein schemes with order 1.

The Theorems 5 and 9 also apply to the family of split-step Euler schemes

$$x^*_\ell=x_{\ell-1}+h_\ell(\alpha f(x^*_\ell,t_\ell)+(1-\alpha)f(x_{\ell-1},t_{\ell-1})), \tag{3.12}$$

$$x_\ell=x^*_\ell+G(x^*_\ell,t_\ell)\Delta w_\ell, \tag{3.13}$$

where $\alpha\in[0,1]$ is a parameter. Unless $\alpha=0$, (3.12) is an implicit deterministic equation in x^*_ℓ. For $\alpha=1$ we obtain the split-step backward Euler scheme (SSBE), which is studied e.g. in [17] for autonomous SDEs. In [17] strong convergence of order $\frac{1}{2}$ is proved under only one-sided Lipschitz conditions and a polynomial growth condition for the drift coefficient. The SSBE is also studied in [24], where it is shown to be effective for inheriting ergodicity in special applications.

From the numerical theory for ordinary differential equations (ODEs) [8,16] it is known that the implicit equation in x,

$$x-z-h(\alpha f(x,t+h)+(1-\alpha)f(z,t))=0 \tag{3.14}$$

possesses a unique solution $x=\chi_\alpha(z,t,h)$ for all $z\in\mathbb{R}^m,t\in\mathscr{J},h$ with $h\alpha\mu\leqslant 1/2$ if f is continuous, has a continuous derivative with respect to x and fulfils the one-sided Lipschitz condition.

$$\langle f(x,t)-f(\tilde{x},t),x-\tilde{x}\rangle\leqslant\mu|x-\tilde{x}|^2 \quad \forall x,\tilde{x}\in\mathbb{R}^m,\ t\in\mathscr{J}. \tag{3.15}$$

Hence, under the above conditions on f, Eq. (3.12) possesses the unique solution $x_\ell^* = \chi_\alpha(x_{\ell-1}, t_{\ell-1}, h_\ell)$ and (3.12), (3.13) can be written as the formally explicit scheme

$$\begin{aligned} x_\ell = x_{\ell-1} &+ h_\ell(\alpha f(\chi_\alpha(x_{\ell-1}, t_{\ell-1}, h_\ell), t_\ell) + (1-\alpha) f(x_{\ell-1}, t_{\ell-1})) \\ &+ G(\chi_\alpha(x_{\ell-1}, t_{\ell-1}, h_\ell), t_\ell)\Delta w_\ell. \end{aligned} \tag{3.16}$$

With

$$\varphi(z, x; t, h) := h(\alpha f(\chi_\alpha(z, t, h), t + h) + (1-\alpha) f(z, t)),$$

$$\psi(z; t, h, y) := G(\chi_\alpha(z, t, h), t + h)y,$$

the split-step Euler schemes are of the general form (3.2). Using this form we are able to verify that (3.12), (3.13) is numerically stable in the mean-square sense and strongly (mean-square) consistent with order $\frac{1}{2}$ if the coefficients f, G are sufficiently smooth.

Proposition 11. *Let f, G be continuous and Lipschitz-continuous with respect to x. Then the split-step Euler methods are numerically stable in the mean-square sense.*

If, additionally, f, G are Hölder continuous with exponent $\frac{1}{2}$ with respect to t, then the split-step Euler methods are strongly (mean-square) consistent with order $\frac{1}{2}$.

Proof. We check assumptions (A1)–(A4) of Theorem 5.

Let f, G be Lipschitz-continuous with respect to x with constants L_f, L_G. Then f trivially fulfils a one-sided Lipschitz condition with $\mu = L_f$. For all $h \leqslant 1/(2\alpha L_f)$ Eq. (3.14) has the unique solution $x = \chi_\alpha(z, t, h)$ and the implicit function χ_α is Lipschitz-continuous with respect to z with a constant $L_\chi := 2(1 + h(1-\alpha)L_f)$. Now we see:

(A1) holds with $L_1 := \alpha L_f L_\chi + (1-\alpha)L_f$ and $L_2 := 0$.
(A2) holds since $x_\ell^* = \chi_\alpha(x_{\ell-1}, t_{\ell-1}, h_\ell)$ is $\mathscr{F}_{t_{\ell-1}}$-measurable.
(A3) holds with $L_3 := L_G L_\chi$.
(A4) holds since

$$\mathbb{E}|G(\chi_\alpha(0, t_{\ell-1}, h_\ell), t_\ell)\Delta w_\ell|^2 \leqslant h_\ell \|G(\chi_\alpha(0, t_{\ell-1}, h_\ell), t_\ell)\| < \infty.$$

Hence, the numerical stability follows by Theorem 5.

Using the Lipschitz properties of the implicit function $\chi_\alpha(z, t, h)$, the strong (mean-square) consistency with order $\frac{1}{2}$ is shown in [33]. □

4. Discretization schemes for index 1 SDAEs

Starting with the Euler Maruyama scheme [21] a wide spectrum of numerical methods for SDEs has been developed. However, first deriving an inherent regular SDE and then applying numerical methods to this special SDE would be a very inefficient procedure for various reasons. In general, one would have to apply a numerical method to solve the constraints for the algebraic variables. It would be much more difficult to exploit special structures and sparseness of the given system.

Furthermore, implicit methods are necessary anyway if the underlying dynamics are stiff. Here we aim at numerical methods for SDAEs that should work directly on the given implicit structure, as in the case of deterministic DAEs.

Not all the discretization schemes for SDEs are suitable for SDAEs as well. Explicit numerical schemes are not suitable for the nonlinear equation solving task involved in SDAEs. The new iterates would not be uniquely determined by an explicit scheme. In this section we will formulate methods for SDAEs that are adaptations of drift-implicit methods used for SDEs. We will formulate them in such a way that the convergence properties of these methods for SDEs are preserved. Nevertheless, the iterates are influenced more critically by computational errors in the constraints.

4.1. The drift-implicit Euler scheme

The SDAE

$$Ax(s)|_{t_0}^{t} + \int_{t_0}^{t} f(x(s),s)\,\mathrm{d}s + \int_{t_0}^{t} G(x(s),s)\,\mathrm{d}w(s) = 0, \quad t \in \mathscr{J}, \tag{4.1}$$

$$Ax(t_0) = Ax^0 \tag{4.2}$$

is discretized by the drift-implicit Euler scheme

$$A\,\frac{x_\ell - x_{\ell-1}}{h_\ell} + f(x_\ell,t_\ell) + G(x_{\ell-1},t_{\ell-1})\,\frac{\Delta w_\ell}{h_\ell} = 0, \quad \ell = 1,\dots,N, \tag{4.3}$$

on the deterministic grid $0 = t_0 < t_1 < \cdots < t_N = T$, where $h_\ell = t_\ell - t_{\ell-1}, \Delta w_\ell = w(t_\ell) - w(t_{\ell-1})$, and x_0 is a given consistent initial value with $Ax_0 = Ax^0$. The Jacobian of (4.3) with respect to the new iterate x_ℓ is $(1/h_\ell)A + f_x'(x_\ell,t_\ell)$, which is nonsingular for sufficiently small stepsizes h_ℓ. Its condition number behaves like $\mathcal{O}(1/h_\ell)$ (see e.g.[12]).

The crucial point for the good properties of this scheme is that the iterates have to fulfil the constraints of the SDAE at the current time-point

$$Rf(x_\ell,t_\ell) = 0.$$

This allows an analogous decoupling procedure as in Section 2 for the continuous problem. Denote

$$u_\ell := Px_\ell, \quad v_\ell := Qx_\ell, \quad \ell = 0,\dots,N.$$

Then the drift-implicit Euler scheme (4.3) with the consistent initial value x_0 for the SDAE (4.1) is equivalent to the composition

$$x_\ell := u_\ell + \hat{v}(u_\ell,t_\ell), \quad \ell = 0,\dots,N,$$

and the following scheme in the differential solution parts u_ℓ:

$$\frac{u_\ell - u_{\ell-1}}{h_\ell} + A^- f(u_\ell + \hat{v}(u_\ell,t_\ell),t_\ell) + A^- G(u_{\ell-1} + \hat{v}(u_{\ell-1},t_{\ell-1}),t_{\ell-1})\,\frac{\Delta w_\ell}{h_\ell} = 0$$

with the initial value $u_0 := Px_0$. This is the drift-implicit Euler scheme applied to the inherent SDE (2.15). In other words, first discretizing, then decoupling leads to the same result as first decoupling and then discretizing for the drift-implicit Euler scheme. Thus, convergence results for (4.3) applied to the SDAE (4.1) can be deduced from convergence results of the drift-implicit Euler scheme applied to SDEs. We can trace back conditions on the coefficients of the inherent SDE to corresponding ones on the coefficients of the original SDAE. In this way all known convergence results for the drift-implicit Euler scheme for SDEs apply to index 1 SDAEs, too.

For a more detailed consideration, including also the numerical stability of the method, we compare the exact solution $\{x_\ell\}$ of (4.3) with the solutions of a scheme perturbed by small errors $\{d_\ell\}$. The measurement of errors in the discrete equations depends on the scaling of these equations. In contrast to ODEs or SDEs, there is no natural scaling of DAEs or SDAEs. Choosing errors in a setting corresponding to that of local truncation errors we consider the system

$$A(\tilde{x}_\ell - \tilde{x}_{\ell-1}) + h_\ell f(\tilde{x}_\ell, t_\ell) + G(\tilde{x}_{\ell-1}, t_{\ell-1})\Delta w_\ell = d_\ell, \quad \ell = 1, \ldots, N, \tag{4.4}$$

or

$$A\frac{\tilde{x}_\ell - \tilde{x}_{\ell-1}}{h_\ell} + f(\tilde{x}_\ell, t_\ell) + G(\tilde{x}_{\ell-1}, t_{\ell-1})\frac{\Delta w_\ell}{h_\ell} = \frac{d_\ell}{h_\ell}, \quad \ell = 1, \ldots, N,$$

with the initial value $\tilde{x}_0$. Denote $\tilde{u}_\ell := P\tilde{x}_\ell$, $\tilde{v}_\ell := Q\tilde{x}_\ell$.

Applying the projector R we obtain the perturbed constraint

$$Rf(\tilde{u}_\ell + \tilde{v}_\ell, t_\ell) = \frac{Rd_\ell}{h_\ell}, \tag{4.5}$$

which, together with the condition $P\tilde{v}_\ell = 0$, implies

$$\tilde{v}_\ell = \hat{v}(\tilde{u}_\ell, t_\ell, Rd_\ell/h_\ell).$$

For the differential parts $\{\tilde{u}_\ell\}$ we obtain the scheme

$$\begin{aligned} &\tilde{u}_\ell - \tilde{u}_{\ell-1} + h_\ell A^- f(\tilde{u}_\ell + \hat{v}(\tilde{u}_\ell, t_\ell, Rd_\ell/h_\ell), t_\ell) \\ &\quad + A^- G(\tilde{u}_{\ell-1} + \hat{v}(\tilde{u}_{\ell-1}, t_{\ell-1}, Rd_{\ell-1}/h_{\ell-1}), t_{\ell-1})\Delta w_\ell = A^- d_\ell, \end{aligned} \tag{4.6}$$

which can be written in the form

$$\tilde{u}_\ell - \tilde{u}_{\ell-1} + h_\ell \hat{f}(\tilde{u}_\ell, t_\ell) + \hat{G}(\tilde{u}_{\ell-1}, t_{\ell-1})\Delta w_\ell = \hat{d}_\ell, \tag{4.7}$$

where

$$\hat{d}_\ell = A^-\{d_\ell - h_\ell d_{f,\ell} - d_{G,\ell-1}\Delta w_\ell\},$$

$$d_{f,\ell} = f(\tilde{u}_\ell + \hat{v}(\tilde{u}_\ell, t_\ell, Rd_\ell/h_\ell), t_\ell) - f(\tilde{u}_\ell + \hat{v}(\tilde{u}_\ell, t_\ell, 0), t_\ell),$$

$$\begin{aligned} d_{G,\ell-1} = {} & G(\tilde{u}_{\ell-1} + \hat{v}(\tilde{u}_{\ell-1}, t_{\ell-1}, Rd_{\ell-1}/h_{\ell-1}), t_{\ell-1}) \\ & - G(\tilde{u}_{\ell-1} + \hat{v}(\tilde{u}_{\ell-1}, t_{\ell-1}, 0), t_{\ell-1}). \end{aligned}$$

Suppose that the perturbations d_ℓ are $\mathscr{F}_{t_\ell}$-measurable with finite second moments and denote

$$\delta_\ell := \|d_\ell\|_{L_2}, \quad \bar{\delta}_\ell := \|\mathbb{E}(d_\ell|\mathscr{F}_{t_{\ell-1}})\|_{L_2},$$

$$\hat{\delta}_\ell := \|\hat{d}_\ell\|_{L_2}, \quad \bar{\hat{\delta}}_\ell := \|\mathbb{E}(\hat{d}_\ell|\mathscr{F}_{t_{\ell-1}})\|_{L_2}.$$

Then $d_{f,\ell}$ is $\mathscr{F}_{t_\ell}$-measurable, $d_{G,\ell-1}$ is $\mathscr{F}_{t_{\ell-1}}$-measurable, and we can estimate

$$\begin{aligned}\hat{\delta}_\ell &\leqslant \|A^-\|(\|d_\ell - h_\ell d_{f,\ell}\|_{L_2} + \|d_{G,\ell-1}\Delta w_\ell\|_{L_2})\\ &\leqslant \|A^-\|((1+L_f L_{\hat{v},d})\delta_\ell + \|d_{G,\ell-1}\|_{L_2}\cdot h_\ell^{1/2})\\ &\leqslant \|A^-\|((1+L_f L_{\hat{v},d})\delta_\ell + L_G L_{\hat{v},d}\|Rd_{\ell-1}\|_{L_2}/h_\ell^{1/2})\\ &=: c_1\delta_\ell + c_2\|Rd_{\ell-1}\|_{L_2}/h_\ell^{1/2},\\ \bar{\hat{\delta}}_\ell &\leqslant \|A^-\mathbb{E}(d_\ell - h_\ell d_{f,\ell}\mid \mathscr{F}_{t_{\ell-1}})\|_{L_2} \leqslant \|A^-\|(\bar{\delta}_\ell + h_\ell\|\mathbb{E}(d_{f,\ell}\mid\mathscr{F}_{t_{\ell-1}})\|_{L_2})\\ &\leqslant \|A^-\|(\bar{\delta}_\ell + L_f L_{\hat{v},d}\|Rd_\ell\|_{L_2}) =: c_3\bar{\delta}_\ell + c_4\|Rd_\ell\|_{L_2}.\end{aligned}$$

Both quantities $\hat{\delta}_\ell, \bar{\hat{\delta}}_\ell$ are more critically affected by perturbations of the constraints. Considering only local discretization errors these critical terms vanish since the values of the exact solution $x(t_\ell)$ fulfil the constraints exactly.

Theorem 9 for the drift-implicit Euler scheme applied to the inherent SDE gives the following estimate for the global errors of the differential components $Px(t_\ell) - P\tilde{x}_\ell$:

$$\begin{aligned}\max_{\ell=1,\ldots,N}\|Px(t_\ell) - P\tilde{x}_\ell\|_{L_2} &\leqslant \hat{S}(\hat{c}h^{\frac{1}{2}} + \max_{\ell=1,\ldots,N}(\hat{\delta}_\ell/h^{\frac{1}{2}} + \bar{\hat{\delta}}_\ell/h))\\ &\leqslant \hat{S}(\hat{c}h^{\frac{1}{2}} + \max_{\ell=1,\ldots,N}(c_1\delta_\ell/h^{\frac{1}{2}} + c_2\|Rd_{\ell-1}\|_{L_2}/(h_\ell h)^{\frac{1}{2}}\\ &\quad + c_3\bar{\delta}_\ell/h + c_4\|Rd_\ell\|_{L_2}/h)).\end{aligned} \tag{4.8}$$

Due to the Lipschitz continuity of the implicit function $\hat{v}$ we obtain, by the composition of solution (2.10):

Corollary 12. *Let the suppositions of Theorem* 2.1 *be fulfilled. Furthermore, let f, G be Hölder continuous with exponent $\frac{1}{2}$ with respect to t with a Hölder constant growing only linearly with x. Then the estimate*

$$\begin{aligned}\max_{\ell=1,\ldots,N}\|x(t_\ell) - x_\ell)\|_{L_2} \leqslant S(\hat{c}h^{\frac{1}{2}} &+ c_1\max_{\ell=1,\ldots,N}\delta_\ell/h^{\frac{1}{2}} + c_3\max_{\ell=1,\ldots,N}\bar{\delta}_\ell/h\\ &+ c_2\max_{\ell=1,\ldots,N}\|Rd_{\ell-1}\|_{L_2}/(h_\ell h)^{\frac{1}{2}} + c_5\max_{\ell=0,\ldots,N}\|Rd_\ell\|_{L_2}/h_\ell)\end{aligned}$$

holds for the global errors $x(t_\ell) - \tilde{x}_\ell$ of the perturbed drift-implicit Euler scheme (4.4).

Proof. The smoothness assumptions on the coefficients f, G of the SDAE (4.1) carry over to corresponding smoothness properties of the coefficients $\hat{f}, \hat{G}$ of the inherent regular SDE. The drift-implicit Euler scheme applied to the inherent regular SDE is strongly (mean-square) consistent with order $\frac{1}{2}$ (compare e.g. [25]) and estimate (4.8) holds for the discretization of the inherent regular SDE. The solution of the SDAE (4.1), and the solution of the perturbed discrete system (4.4) are composed by

$$x(t) := Px(t) + \hat{v}(Px(t), t), \quad \tilde{x}_\ell := P\tilde{x}_\ell + \hat{v}(P\tilde{x}_\ell, t_\ell, Rd_\ell/h_\ell).$$

Due to the Lipschitz property of the implicit function $\hat{v}$ we have

$$|x(t_\ell) - \tilde{x}_\ell| \leqslant (1 + L_{\hat{v}})|Px(t_\ell) - P\tilde{x}_\ell| + L_{\hat{v},d}|Rd_\ell|/h_\ell.$$

In combination with (4.8) we obtain the assertion with $S := (1 + L_{\hat{v}})\hat{S}$ and $c_5 := c_4 + L_{\hat{v},d}/S$ □

4.2. The split-step backward Euler scheme

At the end of Section 3 we considered the split-step backward Euler scheme (SSBE) for SDEs. Here, we intend to construct a scheme for the SDAE (4.1) that should implicitly realize the SSBE for the inherent regular SDE (2.15).

The first step (3.12, $\alpha = 1$) is realized by applying the backward Euler scheme to the deterministic part of the SDAE. However, the second step (3.13) causes more effort than an assignment since we have to force the iterates to fulfil the constraints. Here we give a realization of the SSBE for the index 1 SDAE (4.1), which explicitly uses a projector R along im A:

$$A(x_\ell^* - x_{\ell-1}) + h_\ell f(x_\ell^*, t_\ell) = 0, \tag{4.9}$$

$$A(x_\ell - x_\ell^*) + Rf(x_\ell, t_\ell) + G(x_\ell^*, t_\ell)\Delta w_\ell = 0, \quad \ell = 1, \ldots, N. \tag{4.10}$$

Both Eqs. (4.9), (4.10) are implicit in x_ℓ^* and x_ℓ, respectively. The Jacobian $A + h_\ell f_x'(x_\ell^*, t_\ell)$ of (4.9) has the same structure as for the drift-implicit Euler method. Its condition number behaves like $\mathcal{O}(l/h_\ell)$. The Jacobian $A + Rf_x'(x_\ell, t_\ell)$ of (4.10) is nonsingular with bounded condition number due to the index 1 condition.

The discrete solutions x_ℓ^* and x_ℓ are both forced to fulfil the constraints. That is why (4.9), (4.10) realizes the SSBE scheme for the inherent regular SDE. (4.9),(4.10) show the convergence properties stated in [17] if the inherent regular SDE meets the conditions stated there, where the one-sided Lipschitz condition for the drift-coefficient $\hat{f}$ should be considered on the invariant sub-space im P only. In terms of the original SDAE that means

$$\langle Px - P\tilde{x}, A^-(f(x,t) - f(\tilde{x},t))\rangle \leqslant \mu|Px - P\tilde{x}|^2 \quad \forall x, \tilde{x} \in \mathcal{M}(t),$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product and $\mathcal{M}(t) := \{z \in \mathbb{R}^m\colon\ Rf(z,t) = 0\}$ denotes the constraint manifold.

Alternatively, strong (mean-square) convergence with order $\frac{1}{2}$ is guaranteed if $\hat{f}, \hat{G}$ fulfil the suppositions of the Proposition 11, i.e., f, G are Lipschitz-continuous with respect to x and Hölder continuous with respect to t.

4.3. The trapezoidal rule

The trapezoidal rule is widely used to integrate oscillatory solutions of ODEs. It is A-stable and of order 2 . It is also applied to index 1 DAEs of the form

$$Ax'(t) + f(x(t),t) = 0 \tag{4.11}$$

via the following reformulation: Formally transforming (4.11) to the augmented semi-explicit system

$$x'(t) - y(t) = 0,$$

$$Ay(t) + f(x(t),t) = 0,$$

discretizing the differential equations by the trapezoidal rule

$$\frac{x_\ell - x_{\ell-1}}{h} - \frac{1}{2}\{y_\ell + y_{\ell-1}\} = 0$$
$$Ay_\ell + f(x_\ell, t_\ell) = 0,$$

and reformulating this system to

$$y_\ell := -y_{\ell-1} + 2\,\frac{x_\ell - x_{\ell-1}}{h}, \quad A\left(-y_{\ell-1} + 2\,\frac{x_\ell - x_{\ell-1}}{h}\right) + f(x_\ell, t_\ell) = 0$$

implicitly realizes the trapezoidal rule for the inherent regular ODE. Formally, the augmented system is no longer of index 1 since the constraints $Ay(t) + f(x(t), t) = 0$ are not solvable for the variables y. The components Qy are determined only after a hidden differentiation. However, this does not matter since these components do not enter the implicit formula in x_ℓ. They are of no interest here. Implementing this scheme requires only residuals.

The special member of the family of Euler schemes (3.10) with parameter $\alpha = \frac{1}{2}$,

$$\frac{x_\ell - x_{\ell-1}}{h_\ell} = \frac{1}{2}(f(x_\ell, t_\ell) + f(x_{\ell-1}, t_{\ell-1})) + G(x_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell}\,\Delta w_\ell, \tag{4.12}$$

forms a stochastic counterpart of the trapezoidal rule for the integration of SDEs (3.1). It is of strong order 1/2 like the other Euler schemes.

An adaptation of this scheme to SDAEs, analogous to the deterministic case, would lead to an implicit discretization of the diffusion term. A way out is to create explicit constraints. This can be done by a suitable scaling of (4.1): We scale the system by a nonsingular matrix $\tilde{D}$ such that

$$\tilde{D}A = \begin{pmatrix} \tilde{D}_1 A \\ \tilde{D}_2 A \end{pmatrix} = \begin{pmatrix} \tilde{A}_1 \\ 0 \end{pmatrix} \quad \text{or } \tilde{D}R = \begin{pmatrix} 0 \\ \tilde{R}_2 \end{pmatrix}, \ \operatorname{rank} \tilde{A}_1 = \operatorname{rank} A,$$

and denote analogously

$$\tilde{D}f = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}, \quad \tilde{D}G = \begin{pmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{pmatrix}.$$

Then the trapezoidal rule is realized by

$$\tilde{A}_1 \frac{x_\ell - x_{\ell-1}}{h_\ell} + \frac{1}{2}\{\tilde{f}_1(x_\ell, t_\ell) + \tilde{f}_1(x_{\ell-1}, t_{\ell-1})\} + \tilde{G}_1(x_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell}\Delta w_\ell = 0$$
$$\tilde{f}_2(x_\ell, t_\ell) = 0. \tag{4.13}$$

The iterates fulfil the constraints at the current time-point. Hence, the trapezoidal rule for the inherent SDE is realized. Since the differential equations and the constraints are now decoupled, it is possible to use a different scaling for both parts, which leads to a better conditioned system:

$$\tilde{A}_1(x_\ell, x_{\ell-1}) + \frac{h_\ell}{2}\{\tilde{f}_1(x_\ell, t_\ell) + \tilde{f}_1(x_{\ell-1}, t_{\ell-1})\} + \tilde{G}_1(x_{\ell-1}, t_{\ell-1})\Delta w_\ell = 0$$
$$\tilde{f}_2(x_\ell, t_\ell) = 0. \tag{4.14}$$

The Jacobian of (4.14) with respect to the new iterate is

$$\begin{pmatrix} \tilde{A}_1 + h_\ell \tilde{f}'_{1_x}(x,t)/2 \\ \tilde{f}'_{2_x}(x,t) \end{pmatrix},$$

which is nonsingular for sufficiently small stepsizes and whose condition number is bounded independently of the stepsizes. If the right-hand side of (4.14) is perturbed by vectors $\tilde{d}_\ell$, the factor $1/h_\ell$ in the perturbations of the constraints is avoided. Denote by $\tilde{x}_\ell$ the solution of this perturbed system and let the assumptions of Corollary 12 be fulfilled. Then the estimate

$$\max_{\ell=1,\dots,N} \|x(t_\ell) - \tilde{x}_\ell)\|_{L_2} \leqslant S(\tilde{c}_0 h^{\frac{1}{2}} + \tilde{c}_1 \max_{\ell=1,\dots,N} \tilde{\delta}_\ell / h^{\frac{1}{2}} + \tilde{c}_3 \max_{\ell=1,\dots,N} \bar{\bar{\tilde{\delta}}}_\ell / h),$$

holds, where

$$\tilde{\delta}_\ell := \|\tilde{d}_\ell\|_{L_2}, \bar{\bar{\tilde{\delta}}}_\ell := \|\mathbb{E}(\tilde{d}_\ell \mid \mathscr{F}_{t_{\ell-1}})\|_{L_2}.$$

4.4. The drift-implicit Milstein scheme

In Section 3 we also considered the family of Milstein schemes (3.11) and showed their numerical stability. Compared to the family of Euler schemes (3.10) they possess a higher order of strong (mean-square) convergence, namely order 1. This has to be paid for with the use of the double Itô-integrals and of the Jacobians $(g_j)'_x$ in the scheme.

Our intention is to construct a scheme for the SDAE (4.1) that should realize the drift-implicit Milstein scheme (3.11) with parameter $\alpha = 1$ applied to the inherent SDE (2.15), i.e.,

$$\frac{u_\ell - u_\ell - 1}{h_\ell} + \hat{f}(u_\ell, t_\ell) + \hat{G}(u_{\ell-1}, t_{\ell-1}) \frac{\Delta w_\ell}{h_\ell} - \sum_{j=1}^{m} (\hat{g}'_{j_u} \hat{G})(u_{\ell-1}, t_{\ell-1}) \frac{I_{(j),\ell}}{h_\ell} = 0,$$

where $\hat{G} = (\hat{g}_1, \dots, \hat{g}_m)$, and $I_{(j),\ell} = (I_{j,i;t_{\ell-1},h_\ell})_{i=1}^m$. This is realized by the scheme

$$A \frac{x_\ell - x_{\ell-1}}{h} + f(x_\ell, t_\ell) + G(x_{\ell-1}, t_{\ell-1}) \frac{\Delta w_\ell}{h} - \sum_{j=1}^{m} (g'_{j_x} x'_u A^- G)(x_{\ell-1}, t_{\ell-1}) \frac{I_{(j),\ell}}{h_\ell} = 0, \tag{4.15}$$

since the iterates fulfil the constraints at the current time-point. We point out the explicit use of the inner derivative $x'_u = I + \hat{v}'_u$ and a scaling A^- with $A^- A = P$ in the last term of (4.15). The inner derivative can be expressed as

$$x'_u = I + \hat{v}'_u = I - (A + \lambda R f'_x)^{-1} \lambda R f'_x = I - I + (A + \lambda R f'_x)^{-1} A = (A + \lambda R f'_x)^{-1} A$$

with a free parameter $\lambda \neq 0$. Choosing $\lambda = h$, it may be approximated via

$$x'_u = (A + hRf'_x)^{-1} A = (A + hf'_x)^{-1} A + \mathcal{O}(h)$$

by means of the Jacobian of Newton's method. Scheme (4.15) is closely related to a scheme developed in [26] for the application in circuit simulation, where an approximation to such a scaling is involved.

For problems with multiple Wiener processes ($m > 1$), $m(m-1)/2$ mixed double Itô-integrals have to be approximated in general. The terms involving these mixed double Itô-integrals disappear if the diffusion coefficients are commutative in the sense that

$$[\hat{g}_i, \hat{g}_j] = [A^- g_i, A^- g_j] := \hat{g}'_{i_u}\hat{g}_j - \hat{g}'_{j_u}\hat{g}_i = 0 \quad \forall i \neq j.$$

We remark that this condition depends on the inner derivative x'_u and the scaling A^-, too.

Due to the condition that the iterates fulfil the constraints at the current time-point we obtain an error estimation similar to that in Corollary 12, but with discretization errors of order 1.

5. Application in circuit simulation

In industry, circuit analysis is a standard tool for the design of integrated circuits. One of the most used techniques is the charge-oriented Modified Nodal Analysis (MNA). The equations are generated automatically by combining the network topology, Kirchhoff's Current Law, and the characteristic equations describing the physical behaviour of the network elements. This results in large systems of DAEs, whose special structure was analysed in a number of papers, e.g. [10,13,31]. The increasing scale of integration of electric circuits, among other things, leads to decreasing signal-to-noise ratios. In special applications, where linear noise analysis is no longer satisfactory, transient noise analysis becomes necessary.

Here, we deal with models of thermal noise of resistors and shot noise of *pn*-junctions. Both are modelled as external Gaussian white noise sources in parallel to the original element (see Figs. 1 and 2).

Nyquist's theorem (see e.g. [4,9,32]) states that the current through an arbitrary linear resistor having a resistance R, maintained in thermal equilibrium at a temperature T, can be described as the sum of the deterministic current and a Gaussian white noise process with spectral density

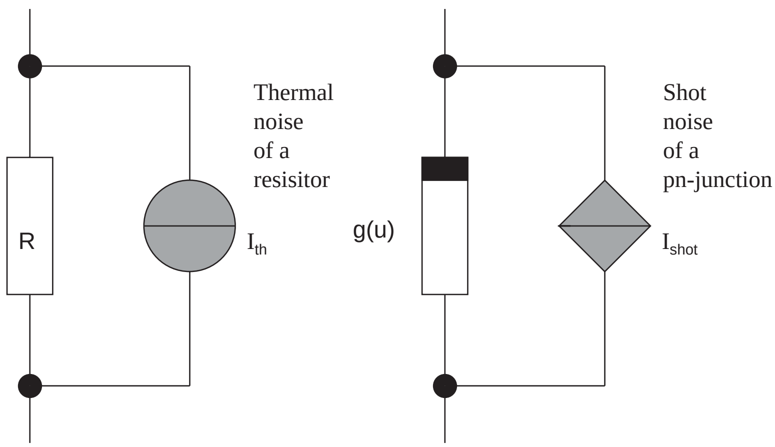

Fig. 2.

$S_{\rm th} := 2kT/R$, where k is Boltzmann's constant. Hence, the additional current is modelled as

$$I_{\rm th} = \sigma_{\rm th} \cdot \xi(t) = \sqrt{\frac{2kT}{R}} \cdot \xi(t),$$

where $\xi(t)$ is a standard Gaussian white noise process. In [32] a thermodynamical foundation to apply this model to mildly nonlinear resistors and reciprocal networks is given.

Shot noise of pn-junctions, caused by the discrete nature of current due to the elementary charge, is also modelled by a Gaussian white noise process. Here the spectral density is proportional to the current I through the pn-junction: $S_{\rm shot} := q|I|$, where q is the elementary charge. If the current through the pn-junction is described by a characteristic $I = g(u)$ in dependence on a voltage u, the additional current is modelled by

$$I_{\rm shot} = \sigma_{\rm shot}(u) \cdot \xi(t) = \sqrt{q|g(u)|} \cdot \xi(t),$$

where $\xi(t)$ is a standard Gaussian white noise process. For a discussion of the model assumptions we refer to [4,9,32].

Now, we consider an electrical network with n_v capacitances, n_R resistances, n_L inductances, $n_V + n_I$ possibly controlled voltage and current sources, and $n_{\rm N}$ additional noise sources. Each element corresponds to a branch connecting two nodes of the network. Given n_e nodes plus the datum node. The network model is determined by its topology, which is represented by means of the incidence matrix $(A_C, A_R, A_L, A_V, A_I, A_N)$, and the characteristic equations of its elements. Using vector-valued characteristics, this extends to multi-ports, too. For a detailed description we refer to [10,13].

Combining Kirchhoff's Current Law and the characteristic equations of the voltage-controlled elements, supplemented by the characteristic equations of inductances and voltage sources and the defining equations for the charges and fluxes, leads to the charge-oriented MNA system with the following structure (see [10,13] for the deterministic case):

$$A_C q' + f_1(e, j_L, j_V, t) + A_N \operatorname{diag}(\sigma(A_N^T e, t))\xi(t) = 0, \tag{5.1}$$

$$\phi' - A_L^T e = 0 \tag{5.2}$$

$$A_V^T e - v_s(e, j_L, t) = 0 \tag{5.3}$$

$$q - q_C(A_C^T e, t) = 0 \tag{5.4}$$

$$\phi - \phi_L(j_L, t) = 0, \tag{5.5}$$

where $f_1(e, j_L, j_V, t) := A_R g(A_R^T e, t) + A_L j_L + A_V j_V + A_I i_s(e, j_L, j_I, t)$, and $q_C, g, \phi_L, v_s, i_s, \sigma$ are given functions. The vector of unknowns describing the system behaviour consists of all node potentials e, the branch currents of current-controlled elements (inductances and voltage sources) j_L, j_V, and the charges q of capacitances and the fluxes ϕ of inductances. ξ denotes an $n_{\rm N}$-dimensional vector of independent standard Gaussian white noise processes. In industry-relevant applications one has to deal with a large number of unknowns as well as of noise sources.

The first block of equations (5.1) means a stochastic integral equation:

$$A_C q(s)|_{t_0}^{t} + \int_{t_0}^{t} f_1(x(s), s)\,\mathrm{d}s + \int_{t_0}^{t} G_1(x(s), s)\,\mathrm{d}w(s) = 0,$$

where $G_1 := A_N \operatorname{diag}(\sigma)$, and w denotes an n_{N}-dimensional Wiener process. With

$$A := \begin{pmatrix} A_C & & \\ & I_{n_L} & \\ & & 0 \end{pmatrix}, \quad G := \begin{pmatrix} A_N \operatorname{diag}(\sigma) \\ 0 \\ 0 \end{pmatrix}. \tag{5.6}$$

Eqs. (5.1)–(5.5) forms a specially structured SDAE of the type (2.1) discussed in the previous sections. We will start discussing the assumptions of Theorem 4 for the SDAE (5.1)–(5.5).

First, we need sufficiently smooth coefficients f, G. They are globally Lipschitz-continuous with respect to x and continuous with respect to t if this is true for the model functions $q_C, r, \phi_L, v_s, i_s, \sigma$.

Second, we need to ensure that the noise-sources do not appear in the constraints, i.e., $\operatorname{im} G(x,t) \subseteq \operatorname{im} A$.

This is guaranteed if $\operatorname{im} A_N \subseteq \operatorname{im} A_C$, or, in terms of the network topology, if there is always a path of capacitances in parallel to a noise source.

Third, we need to ensure that the constraints are globally uniquely solvable for the algebraic variables. This follows if the Jacobian $A + Rf'_x$ is globally bounded invertible, where R is a projector along $\operatorname{im} A$.

Let $Q_C \in \mathbb{R}^{n_e \times n_e}$ be a projector onto $\ker A_C^T$. Then $A_C^T Q_C = 0 \in \mathbb{R}^{n_C \times n_e}$, $Q_C^T A_C = 0 \in \mathbb{R}^{n_e \times n_C}$, and Q_C^T is a projector along im A_C. Hence,

$$R := \begin{pmatrix} Q_C^T & & \\ & 0_{n_L} & \\ & & I \end{pmatrix} \tag{5.7}$$

is a projector along $\operatorname{im} A$. Denote

$$C(u,t) := (q_C)'_u(u,t), \quad G(u,t) := r'_u(u,t), \quad L(j_L,t) := (\phi_L)'_{j_L}(j_L,t).$$

If there are no controlled sources, i.e., $v_s(e, j_L, t) = v_s(t)$, $i_s(e, j_L, j_V, t) = i_s(t)$, we obtain a Jacobian with the following block structure:

$$A + Rf'_x = \begin{pmatrix} A_C & & Q_C^T A_R G A_R^T & Q_C^T A_L & Q_C^T A_V \\ & I_{n_L} & & & \\ & & A_V^T & & \\ I_{n_C} & & -CA_C^T & & \\ & I_{n_L} & & -L & \end{pmatrix}.$$

Now, we assume:

- The matrices $C(u,t), G(u,t), L(j_L,t)$ are symmetric and uniformly positive definite.
- The matrix $Q_C^T A_V$ has full column rank, which means that there are no loops of capacitances and voltage sources.
- The matrix (A_C, A_R, A_V) has full row rank, which means that there are no cut-sets of inductances or current sources.

Then the Jacobian $A + Rf'_x$ is nonsingular and its inverse is uniformly bounded. Following the lines of [10], this result remains true for controlled sources as long as they fulfil certain conditions described there.

Summarizing we have:

Corollary 13. *Suppose that the functions $q_C, r, \phi_L, v_s, i_s, \sigma$ are globally Lipschitz-continuous with respect to the unknown variables and continuous with respect to t, and that the partial derivatives $C(u,t), G(u,t), L(j_L,t)$ are uniformly positive definite. Suppose that there is always a path of capacitances in parallel to a noise source, that there are no loops of capacitances and voltage sources, and no cut-sets of inductances or current sources, and that controlled voltage or current sources fulfil the conditions described in* [10].

Then system (5.1)–(5.5) *has a pathwise unique solution process with properties described in Theorem* 4.

We conclude with a brief discussion of the discretization schemes from Section 4 for the charge-oriented MNA-system (5.1)–(5.5).

The drift-implicit Euler scheme can be implemented straightforwardly. Realizing the SSBE scheme (4.9), (4.10) requires a projector R along $\operatorname{im} A$. Such a projector is given in (5.7) based on a projector Q_C^T along $\operatorname{im} A_C$. Similarly, using a scaling $\tilde{D}_C$ with $\tilde{D}_C A_C = \binom{\tilde{A}_C}{0}$ one obtains a scaling D to implement the trapezoidal rule with splitted residuals as described in Section 4.3. Furthermore, the scaling A^- with $A^- A = P$, which is needed in the Milstein scheme in Section 4.4, is determined by the matrix A from (5.6), basically by A_C already, and thus by the topology of the network.

Since the additional smoothness conditions on the coefficients f resp. G, which are needed to guarantee the numerical stability and consistency of the considered discretization schemes, are transferred to corresponding smoothness conditions on the model functions q_C, r, ϕ_L, v_s, i_s, and σ, respectively, the following corollary holds:

Corollary 14. *Let the suppositions of Corollary* 13 *hold true. Additionally, suppose that the functions $q_C, r, \phi_L, v_s, i_s, \sigma$ are Hölder continuous with exponent* 1/2 *with respect to t.*

Then the drift-implicit Euler scheme (4.3), *the SSBE scheme* (4.9), (4.10) *as well as the trapezoidal rule* (4.13) *applied to system* (5.1)–(5.5) *are numerically stable in the mean-square sense and strongly* (*mean-square*) *convergent with order* 1/2.

Under further smoothness conditions an analogous result ensures that the drift-implicit Milstein scheme (4.15) applied to system (5.1)–(5.5) is strongly (mean-square) convergent with order 1.

6. Conclusions

In contrast to SDEs, in SDAEs the solution has to fulfil constraints. The numerical approximations also have to be forced to do so. In the present paper an approach to the numerical analysis of generally nonlinear DAEs driven by Gaussian white noise is developed. Here, the fact that the leading Jacobian in front of the derivatives is constant is extensively used. It allows a decoupling of the SDAE as well as of its discretization underlying the theoretical results by applying constant projectors, which is compatible to the Itô calculus. The drift-implicit methods discussed in Section 4 work directly on the given structure and are formulated in such a way that the convergence properties of these methods known for SDEs are preserved. The presented approach applies to weakly convergent schemes, too.

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